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Technical Memorandum

A DISCUSSION OF TAYLOR WEIGHTING FOR CONTINUOUS APERTURES

Date: 4 January 1985

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I. INTRODUCTION

This Memorandum is a review of Taylor's original weighting function for continuous apertures. It is presented in some detail in Sections II and III. It is shown that Taylor's beampattern and weighting function can be computed easily by analytically exact formulas. Taylor's beampattern turns out to be the product of a rational function and the beampattern of a uniformly weighted aperture.

Also reviewed is a modification due to Rhodes of Taylor's pattern for the purpose of forcing the weighting function to go zero as a power α of distance from the aperture endpoints. This results in a 5% increase in beamwidth over the beamwidth of Taylor's original pattern if $\alpha = 1$, and a 10% increase if $\alpha = 2$ (for $\bar{n} = 10$; see below). These modifications are discussed in Section IV.

Taylor's original paper [1] derives a symmetric weighting function for a continuous aperture. He does not discuss or even mention its use for arrays of point sensors. His method is essentially an ad hoc, but intuitively sensible, procedure which blends together the desirable characteristics of uniform weighting and the van der Maas weighting into one weighting design. The blending is accomplished by careful specification of the beampattern nulls. The various sidelobe levels do not enter the method's derivation. In other words, the sidelobes are whatever they turn out to be after specification of the nulls.

It is often said that Taylor weighting makes the first few sidelobes near the mainlobe nearly flat; that is, all "near-in" sidelobes have essentially the same amplitude. This statement is erroneous. See Figure 1, for example, where the 9 sidelobes ($\bar{n} = 10$) nearest the mainlobe would all be at -20 dB if the statement were true. Instead, the first sidelobe is at -20 dB and the ninth sidelobe is at (roughly) -25 dB.

It is a useful fact that the beampattern corresponding to Taylor weighting can be computed analytically, without Fourier transforming the weighting function. This can be seen from Taylor's original discussion [1], which is reviewed in this Memorandum. Taylor's original notation is retained here. Appendix A gives a FORTRAN program which computes the beampattern and/or the weighting function using the analytical formulas developed below. In addition, it computes the exact half-power beamwidth.

The aperture is assumed to lie on the p-interval from $-\pi$ to $+\pi$. The weighting function $g(p)$ is related to the far-field beampattern $F(z)$ by

$$F(z) = \int_{-\pi}^{\pi} g(p)\, e^{izp}\, dp. \tag{1}$$

Taylor assumes throughout that $g(p)$ is a real even function. Consequently, $F(z)$ is also an even function of z.

It is well known that

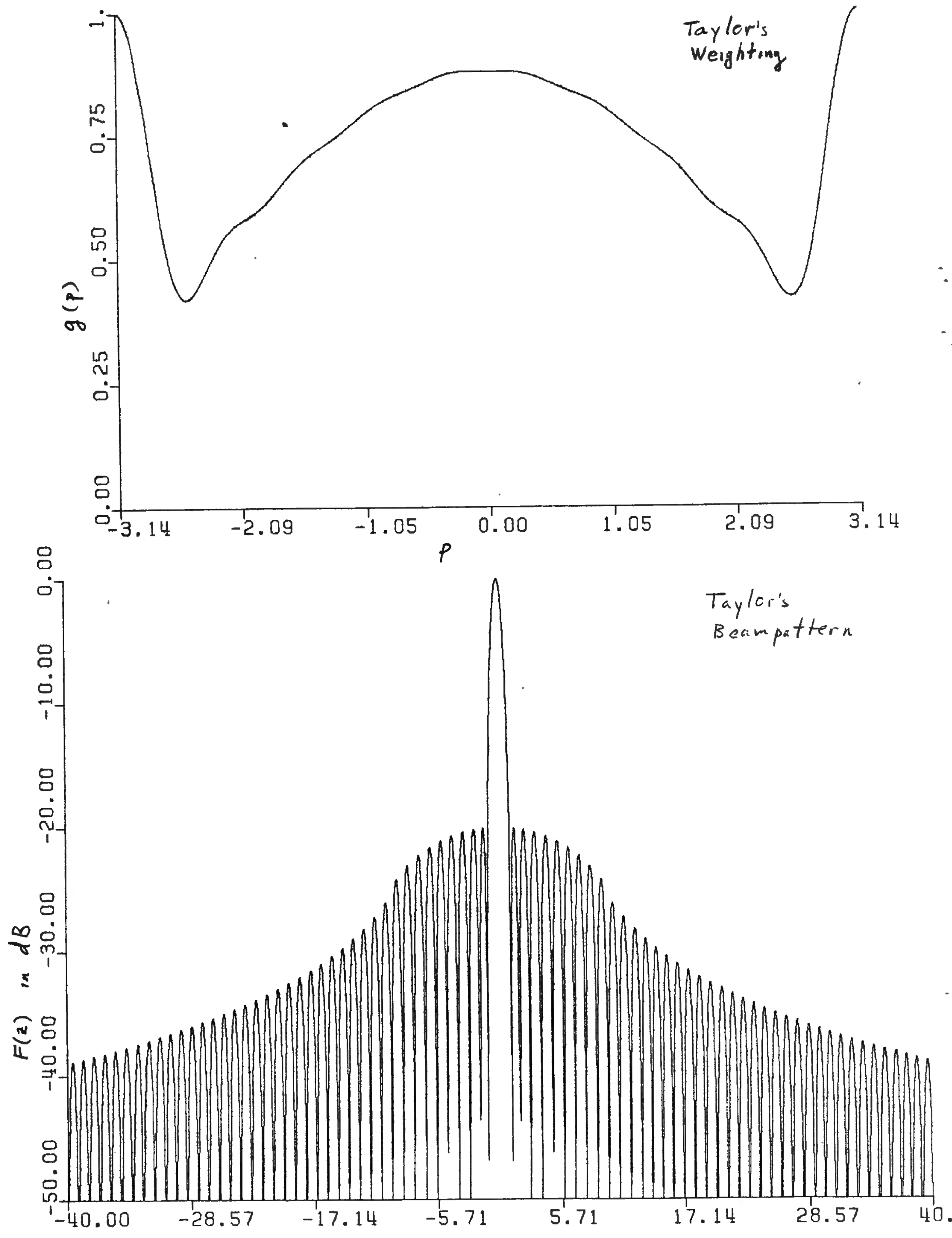

Figure 1. Taylor design for -20dB sidelobe level and $\bar{n} = 10$

$$F(z) = 2\pi \sum_{m=0}^{\infty} \epsilon_m \, F(m) \left(\frac{\sin \pi(z-m)}{\pi(z-m)} + \frac{\sin \pi(z+m)}{\pi(z+m)} \right) \tag{2}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. In other words, knowledge of the integer samples of $F(z)$ implies knowledge of $F(z)$ everywhere. A very different representation of $F(z)$ is the infinite product

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2} \right) \tag{3}$$

where $\{z_1, z_2, \ldots\}$ is a complete list of all the positive zeros of $F(z)$. It is an interesting mathematical fact that these zeros must all lie on the real z-axis. For example, uniform weighting $g(p) = 1/(2\pi)$ gives

$$F(z) = \frac{\sin \pi z}{\pi z} ,$$

whose positive nulls are $\{1,2,3, \ldots.\}$ From (3), then,

$$\frac{\sin \pi z}{\pi z} = \prod_{n = 1}^{\infty} \left(1 - \frac{z}{n^2}\right), \tag{4}$$

a well known identity dating back at least to Euler's time (circa 1750).

By means of his choice of nulls $\{z_n\}$ in the representation (3) of $F(z)$, Taylor sought a beampattern which had a flat envelope near the mainbeam and, for large z, an asymptotic 6 dB/octave decay rate. He also sought by this same means a physically realizable aperture to approximate the physically unrealizable ideal van der Maas function. (It is unrealizable because of the presence of delta function spikes at the aperture end-points, $p = \pm \pi$.) Taylor found a set of nulls which came close to attaining his first objective and which did attain his second objective. The next section is a description of Taylor's nulls.

II. TAYLOR'S NULL SPECIFICATION

Taylor specifies the nulls z_n of his beampattern, starting with $n = \bar{n}$, to be exactly the same as those of the uniform weighting function; that is,

$$z_n = n \qquad \text{for } n = \bar{n}, \bar{n} + 1, \ldots \tag{5}$$

The positive integer $\bar{n}$ is a free parameter which can be chosen as desired. Note that $\bar{n} = 1$ gives exactly uniform shading. Note also that the null list (5) guarantees a 6 dB/octave asymptotic decay rate as $z \rightarrow \infty$. (This follows from (8) below.)

To complete the list of positive nulls for his beampattern, Taylor selects (when $\overline{n} > 1$) the "near-in" nulls to be

$$z_n = \sigma \sqrt{A^2 + (n - \tfrac{1}{2})^2} \quad \text{for } n = 1,2, \ldots, \overline{n} - 1, \tag{6}$$

where

$$\sigma = \overline{n} \,/ \sqrt{A^2 + (\overline{n} - \tfrac{1}{2})^2}$$

$$A = \frac{1}{\pi} \ln \left(R + \sqrt{R^2 - 1}\right)$$

$$R = 10^{|S|/20}$$

S = maximum sidelobe level (in dB).

This choice for the first $\overline{n}-1$ nulls may seem mysterious at first glance, but it is a choice based on the ideal van der Maas function [2], defined by

$$F_0(z, A) = \cos \pi \sqrt{z^2 - A^2}, \quad A>0.$$

It is an interesting mathematical fact that among all beampattern functions $F(z)$ such that

(a) $F(z)$ has a Fourier transform vanishing outside the aperture $-\pi$ to $+\pi$

(b) $|F(z)| \leq 1$ for $|z| \geq A$,

the one with the maximum possible value at $z=0$ is the van der Maas function $F(z) = F_0(z,A)$. The positive nulls of $F_0(z,A)$ are

$$z_n = \sqrt{A^2 + (n - \tfrac{1}{2})^2}, \quad n = 1, 2, 3, \ldots$$

Comparison of these nulls with Taylor's ad hoc null specification (6) shows that Taylor's nulls are related to the van der Maas nulls by a dilation factor σ. The factor σ is chosen to be slightly larger than unity to compensate for the 6 dB/octave decay of the beampattern for $z \geq \overline{n}$. Note that $\overline{n} = \infty$ gives exactly the van der Maas beampattern.

III. TAYLOR'S BEAMPATTERN AND WEIGHTING FUNCTION.

Taylor's beampattern can now be expressed, using (3), as

$$F(z) = \prod_{n=1}^{\overline{n}-1} \left(1 - \frac{z^2}{\sigma^2 \left(A^2 + (n-\tfrac{1}{2})^2\right)}\right) \prod_{n=\overline{n}}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{7}$$

The last expression in (7) can be rewritten, using (4), to give

$$F(z) = \prod_{n=1}^{\bar{n}-1} \left(\frac{1 - \dfrac{z^2}{\sigma^2 \left(A^2 + (n-\frac{1}{2})^2\right)}}{1 - \dfrac{z^2}{n^2}} \right) \frac{\sin \pi z}{\pi z} \tag{8}$$

In this expression, limits must be taken whenever $z = 0, 1, 2, 3, \ldots, \bar{n}-1$ to avoid the indeterminate form 0/0. See Appendix B. Note that Taylor's beampattern is identically the product of a rational function (of degree $\bar{n}-1$ in z^2) and the beampattern of the uniformly weighted aperture, $\sin(\pi z)/\pi z$.

It is clear that Taylor's beampattern can be computed analytically from (8) without computing the weighting function at all. The representation (2) of $F(z)$ is

$$F(z) = 2\pi \sum_{m=0}^{\bar{n}-1} \varepsilon_m \, F(m) \left(\frac{\sin \pi (z-m)}{\pi(z-m)} + \frac{\sin \pi(z+m)}{\pi(z+m)} \right) \tag{9}$$

since $F(n) = 0$ for $n \geq \bar{n}$. This is not as efficient as using (8). However, it does yield an efficient way to compute the weighting function $g(p)$. By Fourier transforming it term by term and using the fact that $F(m) = F(-m)$, we get

$$g(p) = \frac{1}{2\pi} \left\{ F(0) + 2 \sum_{n=1}^{\bar{n}-1} F(m) \cos mp \right\}, \; |p| \leq \pi \tag{10}$$

This is the (spatial) Fourier series of Taylor's weighting function. By computing once and for all the constants $F(0), F(1), \ldots, F(\bar{n}-1)$ using (8), the series (9) can be an efficient formula for computation.

The beamwidth measured between the first nulls is (from (6) with n=1)

$$BW_{NULL} = 2\sigma \sqrt{A^2 + \frac{1}{4}}$$

where σ and A are given as above. An exact formula for the half-power beamwidth is not available. Table 2 gives half-power beamwidths that were computed numerically (using a general purpose subroutine in [3, Chapter 7]). More useful perhaps is the following approximate formula for the half-power beamwidth

$$BW_{3dB} \doteq \left\{\frac{\pi^2}{12} + \frac{1}{2}\sum_{n=1}^{\bar{n}-1}\left(\frac{1}{\sigma^2\,(A^2+(n-\frac{1}{2})^2)} - \frac{1}{n^2}\right)\right\}^{-1/2} \tag{11}$$

To prove (11), note that the asymptotic expansion

$$F(z) = 1 - \left\{\sum_{n=1}^{\bar{n}-1}\frac{1}{\sigma^2\,(A^2+(n-\frac{1}{2})^2)} + \sum_{n=\bar{n}}^{\infty}\frac{1}{n^2}\right\}z^2 \;,\; z \to 0$$

follows immediately from (7). Since

$$\sum_{n=\bar{n}}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\bar{n}-1}\frac{1}{n^2} = \frac{\pi^2}{6} - \sum_{n=1}^{\bar{n}-1}\frac{1}{n^2} \;,$$

we have

$$F(z) = 1 - \frac{\pi^2}{6} + \sum_{n=1}^{\bar{n}-1}\left(\frac{1}{\sigma^2\,(A^2+(n-\frac{1}{2})^2)} - \frac{1}{n^2}\right)z^2 \;,\; z \to 0 \;.$$

Setting $F(z) = 1/2$ and solving for z gives (11).

The accuracy of (11) is good in two limiting cases. As $\bar{n} \to \infty$, $\sigma \to 1$ and (11) becomes

$$BW_{3dB} \doteq \left\{\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{A^2+(n-\frac{1}{2})^2}\right\}^{-1/2} \;. \tag{12}$$

The exact answer for the van der Maas function is

$$BW_{3dB} = 2\left\{A^2 - \left(\frac{1}{\pi}\,\text{arc cosh}\,\left(\frac{1}{2}\cosh \pi A\right)\right)^2\right\}^{1/2}$$

and a comparison with (12) is given in the last row in Table 3. Similarly, for $\bar{n} = 1$, the sum in (11) vanishes and

$$BW_{3dB} \doteq \frac{2\sqrt{3}}{\pi} = 1.103 \text{ radians}$$

which is within 10 percent of the correct answer of $BW_{3dB} = 1.207$ radians for the uniformly weighted aperture.

Table 3 gives the relative error between the approximation (11) and the exact half-power beamwidth for the same entries as in Table 2. It may be concluded from Table 3 that

(a) the approximation (11) is always on the low side of the exact half-power beamwidth, and

(b) the correction required to make (11) exact is a constant factor which depends strongly on the specified sidelobe level and very weakly on $\bar{n}$.

Consequently, a suitable correction factor depending only on specified sidelobe level would make (11) very accurate.

The Taylor weighting function need not always be a positive function. The best way to show this is by example. Consider the case $\bar{n} = 100$ and a sidelobe level of $S = -20$ dB. The weighting function is slightly negative just inside the aperture endpoints (for $p = \pm 3.078761$, for example, Taylor's weight is $-.005519929$). See Figure 2. The Taylor function in practice is nearly always positive for smaller values of $\bar{n}$.

$\bar{n}$	-10dB (A = .578)	-20dB (A= .953)	-30dB (A= 1.32)	-40dB (A=1.69)
5	1.0475	1.3264	1.5526	1.7323
10	1.0009	1.2818	1.5220	1.7262
15	.9851	1.2641	1.5051	1.7126
20	.9771	1.2548	1.4954	1.7036
25	.9724	1.2491	1.4892	1.6975
30	.9692	1.2452	1.4849	1.6932
100	.9581	1.2313	1.4691	1.6761
∞	.9533	1.2252	1.4619	1.6680

Table 2. Exact Taylor half-power beamwidths

$\bar{n}$	-10dB (A=.578	-20dB (A=.953)	-30dB (A=1.32)	-40dB (A=1.69)
5	7.67 %	9.98 %	11.4 %	12.2 %
10	7.57 %	9.91 %	11.3 %	12.2 %
15	7.55 %	9.90 %	11.3 %	12.2 %
20	7.54 %	9.89 %	11.3 %	12.2 %
25	7.55 %	9.89 %	11.3 %	12.2 %
30	7.54 %	9.89 %	11.3 %	12.2 %
100	7.55 %	9.88 %	11.3 %	12.1 %
∞	7.54 %	9.88 %	11.3 %	12.1 %

Table 3. Relative error of approximation (11) to Taylor half-power beamwidths.

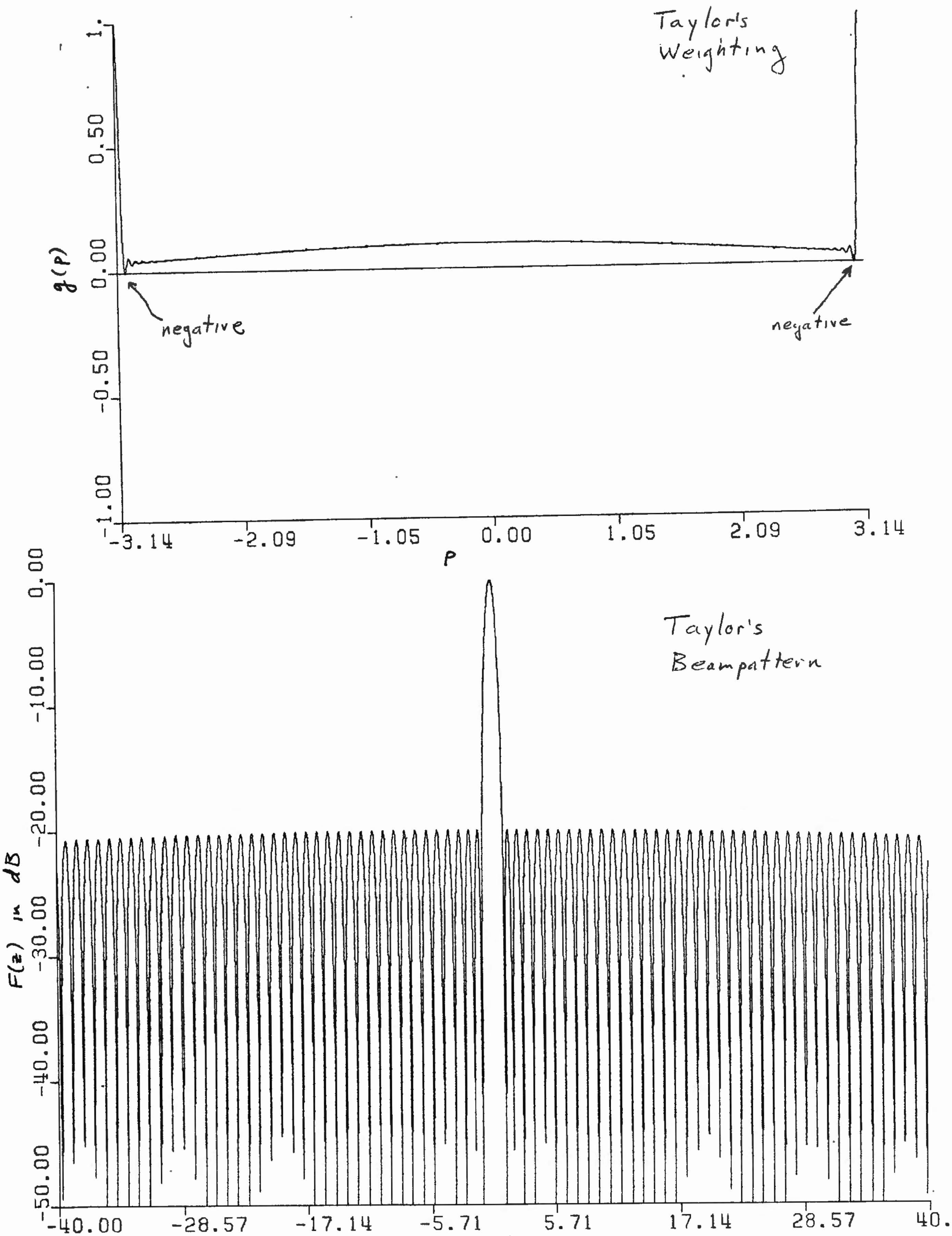

Figure 2. Taylor design for -20 dB sidelobe level and $\bar{n} = 100$.

IV. MODIFICATIONS OF TAYLOR WEIGHTING

Rhodes [4,5] shows that the Taylor weighting function g(p) can be made to go to zero as any power $\alpha > -1$ of distance from the aperture endpoints by altering the position of the nulls in Taylor's function F(z). The general design technique can be viewed as an extension of certain ideas in Taylor's original paper [1], using mathematical methods developed by Rhodes. The most important cases are

1. $\alpha = 0$, which is exactly Taylor's original case; F(z) decays asymptotically at 6 dB per octave.

2. $\alpha = 1$, for which the weighting function goes to zero linearly at the aperture endpoints; F(z) decays asymptotically at 12 dB per octave.

3. $\alpha = 2$, for which the weighting function goes to zero quadratically at the aperture endpoints; F(z) decays asymptotically at 18 dB per octave.

The cases $\alpha = 1$ and $\alpha = 2$ are given explicitly below, after giving the method for any $\alpha > -1$.

A theoretically significant criticism of Rhodes' work is that he does not prove that his technique is mathematically correct. The available theory (due to Paley and Wiener, and to Levinson) provides a proof only for $-1/2 < \alpha < 1/2$ and $\alpha=1$. As Rhodes states [5], "it is not unreasonable to expect that the general theory" is valid for all $\alpha > -1$. In any event, we can proceed to develop the method for all $\alpha > -1$ in a purely formal way, ignoring a theoretical question which may in the end not be of any practical importance. Taylor's original method is, after all, an ad hoc technique and so is Rhodes' generalization of it.

Rhodes' development retains the integer $\bar{n}$ as the breakpoint between the near-in nulls, which are dilated versions of van der Maas' nulls, and the outer nulls, which force the asymptotic decay rate for F(z) to be $6(1+\alpha)$ dB per octave. Consequently, in the limit as $\bar{n} \rightarrow \infty$, the van der Maas function is again obtained for all $\alpha > -1$, just as in Taylor's original design $\alpha = 0$. This means that the desired behavior of the weighting function at the aperture endpoints is confined to small neighborhoods of the aperture endpoints for larger $\bar{n}$. In other words, the weighting function changes rapidly just inside the aperture endpoints for large $\bar{n}$.

The development in [4] is brief and only the case $\alpha = 1$ is given in any detail. His later paper [5] gives enough detail to carry out the general development for $\alpha > -1$. This requires the identity, valid for $\alpha > -1$,

$$T_\alpha(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n + \frac{\alpha}{2})^2}\right) = \Gamma^2 (1 + \alpha/2) \frac{\Gamma(z + 1 - \alpha/2)}{\Gamma(z + 1 + \alpha/2)} \frac{\sin \pi(z - \alpha/2)}{\pi(z - \alpha/2)} \quad (13)$$

It is proved as follows. A special case ($z_1 = z_2 = \alpha/2$ and $z_3 = z$) of a result in [6, Equ. 1.3(4)] gives

$$\frac{\Gamma^2(\alpha/2)}{\Gamma(z + \alpha/2)\,\Gamma(-z + \alpha/2)} = \prod_{n=0}^{\infty}\left(1 - \frac{z^2}{(n + \frac{\alpha}{2})^2}\right).$$

Dividing by the first term in the infinite product, and then using the recurrence formula [6, Equ. 6.1.15] and the reflection formula [6, Equ. 6.1.17] whenever necessary, gives

$$\begin{aligned}
T_\alpha(z) &= \frac{(\alpha/2)^2\,\Gamma^2(\alpha/2)}{[(\alpha/2)^2 - z^2]\,\Gamma(\alpha/2 + z)\,\Gamma(\alpha/2 - z)} \\
&= \frac{\Gamma^2(1 + \alpha/2)}{\Gamma(1 + z + \alpha/2)\,\Gamma(1 - z + \alpha/2)} \\
&= \frac{\Gamma^2(\frac{\alpha}{2} + 1)}{\Gamma(1 + \frac{\alpha}{2} + z)}\,\frac{\Gamma(z - \frac{\alpha}{2})}{\Gamma(z - \frac{\alpha}{2})}\,\frac{1}{\Gamma(1 - (z - \frac{\alpha}{2}))} \\
&= \frac{\Gamma^2(\frac{\alpha}{2} + 1)\,\Gamma(z - \frac{\alpha}{2})}{\Gamma(z + \frac{\alpha}{2} + 1)}\,\frac{\sin \pi(z - \frac{\alpha}{2})}{\pi}
\end{aligned} \tag{14}$$

Multiplying and dividing by $z - (\alpha/2)$ on the right hand side of the last equation yields (13). (We note that above is given without proof by Taylor [1, Equ. (29)].)

Rhodes defines the general Taylor pattern, $F_\alpha(z)$, for $\alpha > -1$ to be

$$F_\alpha(z) = \prod_{n=1}^{\bar{n}-1}\left(1 - \frac{z^2}{\sigma_\alpha^2(A^2 + (n - \frac{1}{2})^2)}\right)\prod_{n=\bar{n}}^{\infty}\left(1 - \frac{z^2}{(n + \alpha/2)^2}\right) \tag{15}$$

where

$$\sigma_\alpha = (\bar{n} + \tfrac{\alpha}{2}) \,/\sqrt{A^2 + (\bar{n} - \tfrac{1}{2})^2} \tag{16}$$

and A is the same as given above (just after (6)). Note that for $\alpha = 0$ the function $F_0(z)$ is exactly Taylor's original function $F(z)$. The analog of (8) for general α is

$$F_\alpha(z) = \prod_{n=1}^{\bar{n}-1} \left(\frac{1 - \dfrac{z^2}{\sigma_\alpha^2 (A^2 + (n - \frac{1}{2})^2)}}{1 - \dfrac{z^2}{(n + \frac{\alpha}{2})^2}} \right) T_\alpha(z) \tag{17}$$

as is clear from (15) and (13). As $z \to \infty$, the rational function of degree $\bar{n}$-1 in z^2 in (17) approaches a constant and $T_\alpha(z)$ is asymptotic to a constant (depending only on α) times $1/|z|^{1+\alpha}$. The asymptotic decay rate of $F_\alpha(z)$ is therefore $6(1 + \alpha)$ dB per octave. In addition, the asymptotic decay rate means that $F_\alpha(z)$ has a Fourier transform vanishing outside the aperture $[-\pi, \pi]$ for every $\alpha > -1$ and $\bar{n} \geq 1$.

For $\alpha > -1$ define the "sampling functions"

$$G_n^{(\alpha)}(z) = c_n(\alpha) \frac{\Gamma(z - \frac{\alpha}{2} + 1)}{\Gamma(z + \frac{\alpha}{2})} \frac{\sin \pi(z - \frac{\alpha}{2})}{\pi(z^2 - (n + \frac{\alpha}{2})^2)} \tag{18}$$

where

$$c_n(\alpha) = \begin{cases} (-1)^n (2n + \alpha)(n + \alpha)/n! & , \text{ if } \alpha \neq 0 \\ 1 & , \text{ if } \alpha = 0,\ n = 0 \\ (-1)^n 2 & , \text{ if } \alpha = 0,\ n = 1,2,3,\ldots \end{cases}$$

Each $G_n^{(\alpha)}(z)$ is an even function of z. These functions are essentially the Lagrange interpolating functions for the points $\{\pm(n + \alpha/2);\ n = 0,1,2,\ldots\}$. More precisely, the only nulls of $G_n^{(\alpha)}(z)$ are of the form $\pm(n + \alpha/2)$ and, furthermore,

$$G_n^{(\alpha)}(\pm(m + \tfrac{\alpha}{2})) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases} \tag{19}$$

The functions $G_n^{(\alpha)}(z)$ are derived using methods due originally to Paley and Wiener.

The open theoretical question mentioned earlier in this section concerns the completeness of the sampling functions (18) with respect to all even aperture-limited functions. As stated already, it is known that $G_n^{(\alpha)}$ are complete for $-1/2 < \alpha < 1/2$ and for $\alpha = 1$. For other values of $\alpha > -1$, nothing is known. Proceeding on the <u>assumption</u> that $G_n^{(\alpha)}$ are complete for all $\alpha > -1$, it follows $F_\alpha(z)$ can be expanded in the form

$$F_\alpha(z) = \sum_{n=0}^{\infty} a_n\, G_n^{(\alpha)}(z)$$

for some constants a_n. From (19) it follows that $a_n = F_\alpha(n + (\alpha/2))$ for all n. From (15) it follows that $a_n = 0$ for $n \geq \bar{n}$. Therefore,

$$F_\alpha(z) = \sum_{n=0}^{\bar{n}-1} F_\alpha(n + \tfrac{\alpha}{2})\, G_n^{(\alpha)}(z) \ , \tag{20}$$

which generalizes (9).

Denote by $g_\alpha(p)$ the weighting function corresponding to $F_\alpha(z)$. As just stated, $g_\alpha(p)$ vanishes outside the aperture $[-\pi, \pi]$. Question: Does $g_\alpha(p)$ go to zero as the power $\alpha > -1$ of distance from the aperture endpoints? Taylor [1] proves that any even function with this endpoint behavior has a Fourier transform whose nulls are asymptotic to $\pm(n + (\alpha/2))$, but he does NOT prove the converse. Consequently, although $g_\alpha(p)$ is even and has a Fourier transform with the proper null locations, this is not necessarily sufficient to answer the question in the affirmative. However, taking the term by term Fourier transform of (20) gives the expansion

$$g_{(\alpha)}(p) = \sum_{n=0}^{\bar{n}-1} F_\alpha(n + \tfrac{\alpha}{2})\, H_n^{(\alpha)}(p) \tag{21}$$

where for n = 0, 1,2, ...

$$H_n^{(\alpha)}(p) = \begin{cases} (2\cos\frac{p}{2})^\alpha \displaystyle\sum_{r=0}^{n} (-1)^{n-r} \frac{\varepsilon_r}{2} \frac{(\alpha)_{n-r}}{(n-r)!} \cos(rp) \, , & |p| < \pi \\ 0 \, , & |p| \geq \pi \end{cases} \tag{22}$$

where $\varepsilon_0 = 1$ and $\varepsilon_r = 2$ for $r > 0$, and

$$(\alpha)_k = \begin{cases} 1 & \text{for all } \alpha, \text{ if } k = 0 \\ (\alpha + 1) \ldots (\alpha + k - 1) & \text{for all } \alpha \ , \text{ if } k > 0. \end{cases}$$

Since each of the functions $H_n^{(\alpha)}(p)$ has the correct endpoint behavior, $g_\alpha(p)$ must also have this same behavior.

Just as in Taylor's original case, both the aperture function $g_\alpha(p)$ and the beampattern function $F_\alpha(z)$ can be computed independently of each other using the analytically exact formulas (21) and (17), respectively, for any $\alpha > -1$. Appropriate approximations near the points $\pm(n + \alpha/2)$ analogous to those developed in Appendix B for $\alpha = 0$, are necessary for computing $F_\alpha(z)$ using (17). Developing these approximations should not present any mathematical difficulties.

The three cases α = 0, 1, 2 are now given explicitly. Fortran programs implementing these three cases should be easy to write. The sampling functions are

$$G_n^{(0)}(z) = (-1)^n \frac{\epsilon_n}{2} \frac{2\, z \sin \pi z}{\pi\, (z^2 - n^2)} \tag{23}$$

$$G_n^{(1)}(z) = (-1)^{n+1} \frac{(2n+1) \cos \pi z}{\pi\, (z^2 - (n+\frac{1}{2})^2)} \tag{24}$$

$$G_n^{(2)}(z) = (-1)^{n+1} \frac{2(n+1)^2 \sin \pi z}{\pi\, z\, (z^2 - (n+1)^2)} \;. \tag{25}$$

and the corresponding aperture basis functions are

$$H_n^{(0)}(p) = \frac{\epsilon_n}{2} \cos np \tag{26}$$

$$H_n^{(1)}(p) = \cos\, (n+\tfrac{1}{2})p \tag{27}$$

$$H_n^{(2)}(p) = (-1)^n + \cos\, (n+1)p \;\;. \tag{28}$$

It should be noted that (23) and (26) are, within a scale factor, identical to (9) and (10), respectively. In all cases, the aperture function $g_\alpha(p)$ is computed from (21). Consequently, the only potential difficulty is computing the constants $F_\alpha(n + \alpha/2)$ for is $n = 0, 1, \ldots, \overline{n} -1$. Fortunately, for $\alpha = 0$, 1, and 2, these constants are easy to compute using (17) since the following identities hold:

$$T_0(z) = \frac{\sin \pi z}{\pi z} \tag{29}$$

$$T_1(z) = \frac{\cos \pi z}{1 - 4z^2} \tag{30}$$

$$T_2(z) = \frac{\sin \pi z}{\pi z\, (1-z^2)} \;\;. \tag{31}$$

The price paid for the desired end effects is an increase in the beamwidth over the beamwidth of Taylor's original weighting function. The beamwidth measured to the first null is, for all $\alpha > -1$,

$$BW_{NULL} = \sigma_\alpha \sqrt{A^2 + \frac{1}{4}}$$

where σ_α is given by (16) above. This gives exactly, for fixed $\overline{n}$,

$$\frac{BW_{NULL}\ (\text{for any } \alpha)}{BW_{NULL}\ (\text{for } \alpha = 0)} = \frac{\sigma_\alpha}{\sigma_0} = 1 + \frac{\alpha}{2\overline{n}} \;.$$

This means that for $\overline{n} = 10$, the beamwidth measured between nulls is 5% larger for $\alpha = 1$ and 10% larger for $\alpha = 2$ than for Taylor's original $\alpha = 0$ beampattern. It is anticipated that approximately the same percentage increases occur in the half-power beamwidths.

A different modification to Taylor's nulls can be utilized to produce asymmetric beampatterns using complex valued aperture functions g(p). This is described in [8] for an application in radar to minimize ground clutter. The magnitude of the aperture function turns out to be an even function, while the phase of the aperture function turns out to be odd.

V. CONCLUSIONS

Taylor weighting can be modified to force the weighting function to go zero as any power $\alpha > -1$ of distance from the aperture endpoints. Taylor's original weighting ($\alpha = 0$) results in a pedestal, while for $\alpha = 1$ the weighting function goes to zero linearly as in a cosine window, and for $\alpha = 2$ the weighting function behaves like a cosine-squared window at the aperture endpoints. The endpoint effect is achieved for a modest increase in the mainlobe beamwidth.

Appendix A.

```
c       COMPUTE TAYLOR'S CONTINUOUS SHADING FUNCTION AND TRANSFER FUNCTION
c       INPUT REQUIREMENT: 1 .LE. NBAR
c                        : THE SPATIAL APERTURE LIES FROM -PI TO +PI
c       ARGUMENT DEFINITIONS:
c               x  = abscissas for sampling the shading function
c               s  = shading function values
c               ns = number of s samples; none computed if ns = 0
c               k  = abscissas for sampling the transfer function
c               f  = transfer function values
c               nk = number of f samples; none computed if nk = 0
c               nbar= the first nbar-1 zeros of f are those of van der Maas
c               db = sidelobe level in db of limiting Dolph-Chebyshev array
c               fm = coefficients of the shading function cosine series
c               bw3db = -3 dB beamwidth; not computed if bw3db is set to -1.
c       DIMENSION LIMITS:
c           x and s must be dimensioned at least max(ns,+1)
c           k and f must be dimensioned at least max(nk,+1)
c           fm must be dimensioned at least nbar
c       TECHNICAL NOTES:
c              NBAR=1 GIVES THE UNIFORM SHADING FUNCTION
c              NBAR=INFINITY GIVES DOLPH-CHEBYSHEV SHADING
c              FIRST BEAMPATTERN NULL = SIGMA * SQRT(A**2+.25)
c              THE COSINE SERIES FOR S HAS DEGREE EXACTLY NBAR-1
c       PROGRAMMER: R. L. STREIT, NUSC, DECEMBER 21, 1984.
c               LAST REVISION: JANUARY 11, 1985
c
        subroutine taylor(x,s,ns,k,f,nk,nbar,db,fm,bw3db)
        double precision x(1),s(1),k(1),f(1),db,fm(1),xpt,a,sigma,q,pi,g,
     +          zeroin,ap,bp,tol,bw3db
        data pi/3.14159265358979d0/
        a=10.0d0**abs(db/20.0d0)
        a=dlog(a+sqrt(a*a-1.0d0))/pi
        sigma=nbar/sqrt(a*a+(nbar-.5d0)*(nbar-.5d0))
        nbar1=nbar-1
        fm(1)=1.0d0
        if(nbar.eq.1)go to 15
        do 10 i=2,nbar
        xpt=i-1
        fm(i)=q(xpt,a,nbar1,sigma)
 10     continue
 15     if(ns.le.0)go to 25
        do 20 i=1,ns
        s(i)=0.0d0
        if(abs(x(i)).gt.(pi+1.d-7))go to 20
        s(i)=g(x(i),fm,nbar)
 20     continue
 25     if(nk.le.0)go to 35
        do 30 i=1,nk
        f(i)=q(k(i),a,nbar1,sigma)
 30     continue
 35     if(bw3db.lt.0.0d0)go to 40
        ap=0.0d0
        bp=sigma*sqrt(a*a+.25d0)
        tol=1.0d-10
        bw3db=2.0d0*zeroin(ap,bp,q,tol,a,nbar1,sigma)
 40     continue
        return
        end
        double precision function q(z,a,nbar1,sigma)
        double precision pi,piz,zn,z,a,sigma
        data pi/3.14159265358979d0/
```

```
      do 10 k=0,nbar1
      if(abs(z-k).lt.1.0d-4)go to 30
 10   continue
      g=1.0d0
      if(nbar1.eq.0)go to 21
      do 20 n=1,nbar1
      zn=(z/sigma)**2/(a*a+(n-.5d0)**2)
      g=g*(1.0d0-zn)/(1.0d0-(z/n)**2)
 20   continue
 21   piz=pi*z
      g=(sin(piz)/piz)*g
      return
 30   if(k.gt.0)go to 50
      g=1.0d0
      if(nbar1.eq.0)go to 41
      do 40 n=1,nbar1
      zn=(z/sigma)**2/(a*a+(n-.5d0)**2)
      g=g*(1.0d0-zn)/(1.0d0-(z/n)**2)
 40   continue
 41   piz=pi*z
      g=(1.0d0-piz*piz*(1.0d0-piz*piz/20.d0)/6.0d0)*g
      return
 50   g=1.0d0
      do 60 n=1,nbar1
      if(n.eq.k)go to 60
      zn=(z/sigma)**2/(a*a+(n-.5d0)**2)
      g=g*(1.0d0-zn)/(1.0d0-(z/n)**2)
 60   continue
      zn=(z/sigma)**2/(a*a+(k-.5d0)**2)
      g=g*(1.0d0-zn)
      piz=pi*(z-k)
      g=(1.0d0-piz*piz*(1.0d0-piz*piz/20.d0)/6.0d0)*g
      g=g*(-1)**(k+1)*(k/(z+z*z/k))
      return
      end
      double precision function g(p,fm,nbar)
      double precision p,twopi,fm(1)
      data twopi/6.2831853071795864 77d0/
      g=0.0d0
      if(nbar.eq.1)go to 20
      do 10 i=2,nbar
      g=g+fm(i)*cos((i-1)*p)
 10   continue
 20   g=(fm(1)+g+g)/twopi
      return
      end
c
c     Compute a zero of a real function f in the interval [ax, bx].
c     Double precision version of program on pp. 164-166 of "Computer
c     Methods for Mathematical Computations," by G.E. Forsythe,
c     M.A. Malcolm, and C.B. Moler, Prentice-Hall, 1977, but slightly
c     altered for use in computing Taylor's half-power beamwidth.
      double precision function zeroin(ax,bx,f,tol,adum,nbar1,sigma)
      double precision ax,bx,f,tol,adum,sigma
      double precision a,b,c,d,e,eps,fa,fb,fc,tol1,xm,p,q,r,s
      eps=1.0d0
 10   eps=eps/2.0d0
      tol1=1.0d0+eps
      if(tol1.gt.1.0d0)go to 10
      a=ax
      b=bx
```

```
      fa=f(a,adum,nbar1,sigma)-.5d0
      fb=f(b,adum,nbar1,sigma)-.5d0
20    c=a
      fc=fa
      d=b-a
      e=d
30    if(abs(fc).ge.abs(fb))go to 40
      a=b
      b=c
      c=a
      fa=fb
      fb=fc
      fc=fa
40    tol1=2.0d0*eps*abs(b)+.5d0*tol
      xm=.5d0*(c-b)
      if(abs(xm).le.tol1)go to 90
      if(fb.eq.0.0d0)go to 90
      if(abs(e).lt.tol1)go to 70
      if(abs(fa).le.abs(fb))go to 70
      if(a.ne.c)go to 50
      s=fb/fa
      p=2.0d0*xm*s
      q=1.0d0-s
      go to 60
50    q=fa/fc
      r=fb/fc
      s=fb/fa
      p=s*(2.0d0*xm*q*(q-r)-(b-a)*(r-1.0d0))
      q=(q-1.0d0)*(r-1.0d0)*(s-1.0d0)
60    if(p.gt.0.0d0)q=-q
      p=abs(p)
      if((2.0d0*p).ge.(3.0d0*xm*q-abs(tol1*q)))go to 70
      if(p.ge.abs(0.5d0*e*q))go to 70
      e=d
      d=p/q
      go to 80
70    d=xm
      e=d
80    a=b
      fa=fb
      if(abs(d).gt.tol1)b=b+d
      if(abs(d).le.tol1)b=b+sign(tol1,xm)
      fb=f(b,adum,nbar1,sigma)-.5d0
      if((fb*(fc/abs(fc))).gt.0.0d0)go to 20
      go to 30
90    zeroin=b
      return
      end
```

Appendix B. Calculation of F(z)

For some small number $\epsilon > 0$, say $\epsilon = 10^{-4}$, define

$$s(z,\, k) = \begin{cases} \dfrac{\sin \pi z}{\pi(z-k)}, & |z-k| > \epsilon > 0 \\[2ex] (-1)^k \left[1 - \dfrac{\pi^2}{6}(z-k)^2 + \dfrac{\pi^4}{120}(z-k)^4 - \dots\right], & |z-k| < \epsilon. \end{cases}$$

Now, if $|z-k| > \epsilon$ for $k = 0,1,2,\dots,\bar{n}-1$, compute $F(z)$ exactly as in (8). If $|z-k| \leq \epsilon$ for $k=0$, then compute

$$F(z) = s(z,0) \prod_{n=1}^{\bar{n}-1} \left(\frac{1 - \dfrac{z^2}{\sigma^2 \left(A^2 + (n - \frac{1}{2})^2\right)}}{1 - \dfrac{z^2}{n^2}} \right).$$

If $|z-k| \leq \epsilon$ for $k=1,\dots,\bar{n}-1$, then compute

$$F(z) = s(z,k)\, \frac{(-k)\left(1 - \dfrac{1}{\sigma^2 \left(A^2 + (k - \frac{1}{2})^2\right)}\right)}{z\left(1 + \frac{z}{k}\right)} \prod_{\substack{n=1 \\ n \neq k}}^{\bar{n}-1} \left\{ \frac{1 - \dfrac{z^2}{\sigma^2\left(A^2 + (n - \frac{1}{2})^2\right)}}{1 - \dfrac{z^2}{n^2}} \right\}.$$

Use of these formulae eliminates all indeterminate 0/0 forms that arise during actual computation using (8).

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